Curve interpolation based on Catmull-Clark subdivision scheme*

ZHANG Jingqiao[1] and WANG Guojin[2]**

(State Key Laboratory of CAD&CG, Institute of Computer Images and Graphics, Zhejiang University, Hangzhou 310027, China)

Received June 11, 2002; revised August 8, 2002

Abstract An efficient algorithm for curve interpolation is proposed. The algorithm can produce a subdivision surface that can interpolate the predefined cubic B-spline curves by applying the Catmull-Clark scheme to a polygonal mesh containing "symmetric zonal meshes", which possesses some special properties. Many kinds of curve interpolation problems can be dealt with by this algorithm, such as interpolating single open curve or closed curve, a mesh of nonintersecting or intersecting curve. The interpolating surface is C^2 everywhere excepting at a finite number of points. At the same time, sharp creases can also be modeled on the limit subdivision surface by duplicating the vertices of the tagged edges of initial mesh, i.e. the surface is only C^0 along the cubic B-spline curve that is defined by the tagged edges. Because of being simple and easy to implement, this method can be used for product shape design and graphic software development.

Keywords: curve interpolation, B-spline, subdivision scheme.

The research of computer aided design and modeling of the curve/surface has become very active over the past few years[1~5]. Especially, the recursive subdivision method has attracted more and more attention although the research on it started more than 20 years ago[6~9] and has already been widely applied to 3D surface modeling, multiresolution and computer graphics. According to their relation with the given geometric information, subdivision schemes can be categorized into two distinct classes, namely, approximating schemes and interpolating schemes. Methods of Doo-Sabin[8] and Catmull-Clark et al.[9] exemplify the former. The schemes of Loop[1)], Peters and Reif[10] and Qin et al.[11] also belong to the class of approximating. The most well-known interpolation-based subdivision scheme is the "butterfly" algorithm proposed by Dyn et al.[12] and Zorin et al.[13] presented an improved interpolatory subdivision scheme. A variational approach for interpolatory refinement has been proposed by Kobbelt[14], and a research using the subdivision surface to solve interpolation problems was carried out by Nasri[15], who constructed subdivision surfaces based on Doo-Sabin scheme to interpolate vertices of polygonal meshes and their normal. In 1993 Halstead[16] proposed the method for constructing a smooth surface, which interpolates the given data set.

It is an essential tool and the pursued purpose in surfaces modeling to interpolate the given curves. For instance, in product shape design one can use some suitable methods to generate a surface to interpolating predefined curves such as feature lines. With the traditional method, also known as lofting, interpolated curves are regarded as cross-sectional curves through which a skinning surface or sweeping surface passes. But during the skinning process, some knots will be removed to avoid generating huge amounts of control points. So the resulting B-spline surface does not interpolate but approximate cross-sectional curves. Furthermore, in general, sweeping surfaces cannot be represented as NUBRS. As the research of subdivision surfaces going deeply, Nasri[17] proposed a method that uses Doo-Sabin subdivision scheme to modeling subdivision surfaces for interpolating given curves. This method, however, can only interpolate quadric B-spline curves. In fact, cubic B-spline curves are more popular in surface modeling. Consequently, the question as to how to construct a subdivision surface to interpolate cubic curves arises. A new method named "combined subdivision" was proposed by Levin[18,19], in which the goal of interpolation can be achieved through changing the rules of subdivision near the interpolated curves that can be given in any parametric representation. But during the subdivision

* Supported by the National Natural Science Foundation of China (Grant No. 60173034) and the Major State Basic Research Development Program of China (Grant No. G1998030600)

** To whom correspondence should be addressed. E-mail: wgj@math.zju.edu.cn or jqzhang@zju.edu.cn

1) Loop, C. Smooth subdivision surfaces based on triangles. Master's thesis, Department of Mathematics, University of Utah, 1987

process, the computation of some necessities must depend on the parameterization of given curves, so complicated algebraic transformation is often needed.

In view of some disadvantages of existing subdivision surface modeling schemes used for curve interpolation, we propose a scheme for interpolating cubic B-spline curves with Catmull-Clark subdivision surface. We need only guarantee the symmetry of polygons on both sides of control polygon edges of interpolated curves. During the process of subdivision, we do not need to modify the subdivision rules near the interpolation curves. Thus the process is convergent and the limit surface is C^2 everywhere except for a finite number of points. Moreover, sharp creases can also be modeled on the limit subdivision surface by duplicating the vertices of the tagged edges of initial mesh, i.e. the surface is only C^0 along the cubic B-spline curve that is defined by the tagged edges. This scheme can deal with many kinds of curve interpolation problems, such as interpolating single open curve or closed curve, and a mesh of nonintersecting or intersecting curves. It is no doubt that our scheme will widen applications of Catmull-Clark subdivision surfaces in the fields of product shape design and computer animation etc..

1 Catmull-Clark subdivision scheme

In 1978, Catmull and Clark[9] generalized bicubic B-spline surfaces over arbitrary topology. The limit surface converges to a bicubic B-spline surface except at a finite number of extraordinary points. The scheme starts with an initial polygonal mesh. Then the rules of new mesh generation consist of two parts, one locating new vertices, the other constructing topological structure.

(i) Locating new vertices

New face points: the average of all the old vertices defining the face.

New edge points: the average of two old vertices defining the edge and two new face points of the two faces sharing the edge.

New vertex points: the average $\frac{n-2}{n}v + \frac{E}{n} + \frac{F}{n}$, where E is the average of the vertices sharing an edge with the old vertex v, F the average of new face points of all faces adjacent to the old vertex v, and n the number of the edges incident on the vertex.

(ii) Constructing topological structure

Connecting each new face point to the new edge points of the edges defining the old face.

Connecting each new vertex point to the new edge points of all old edges incident on the old vertex.

Note that all faces are quadrilaterals after first application of the Catmull-Clark subdivision step.

2 Symmetric zonal mesh

In this section, we define a kind of mesh possessing special properties. So one can predict the curve interpolated by the limit surface of the mesh.

Definition 1. Given a line segment AB, then Chebychev points on AB is given by

$$C_i = \frac{(1+\beta_i)A + (1-\beta_i)B}{2}, \quad i = 1, 2, \cdots, k,$$

where

$$\beta_i = \frac{\cos\frac{(2i-1)\pi}{2k}}{\cos\frac{\pi}{2k}}.$$

Obviously, $C_1 = A$, $C_k = B$, at the same time the points C_i and C_{k+1-i} ($i = 1, 2, \cdots, k$) are symmetric about the midpoint of the line segment AB.

Definition 2. Given two polygons sharing only one edge, called center edge, and having the same number of vertices. The two polygons are called symmetric about their center edge if the corresponding vertices are symmetric about the Chebychev points on the center edge.

Note that two edges, one for each polygon respectively, which intersect at the endpoint of the center edge, are collinear. Fig. 1 shows an example, where the polygons P and Q are symmetric about their center edge C_1C_5, p_i and q_i are the corresponding vertices and symmetric about the points C_i ($i = 1, 2, \cdots, 5$).

Definition 3. A structure, consisting of four polygons sharing a common vertex, is called a symmetric polygon group, if (i) adjacent polygons share only one edge; (ii) two adjacent polygons are symmetric about their center edge, and the rest are also symmetric about their center edge.

The common vertex of the four polygons is called center vertex of the symmetric polygon group. In Fig.2, P_i and $Q_i (i = 1, 2)$ are four polygons, where P_1 and Q_1 are symmetric about the edge C_1C_5, P_2 and Q_2 are symmetric about the edge C_5C_6, and C_5 is the center vertex.

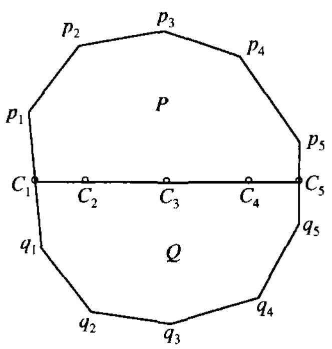


Fig. 1. Two polygons that are symmetric about their center edge.

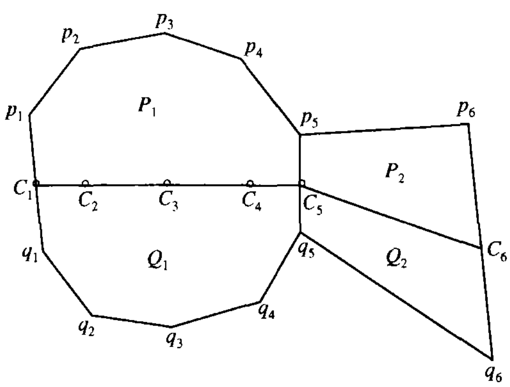


Fig. 2. Symmetric polygon group and its center.

Definition 4. A symmetric polygon group is called a joint polygon group, if (i) polygons in the group are all quadrilaterals, and (ii) any two polygons are symmetric about their center edge.

In Fig.3, we say the edges f_1f_4 and f_2f_3 are opposite, while the edges f_1f_2 and f_2f_3 are adjacent. The vertex v is the center vertex. The edges $e_iv (i = 1, 2, 3, 4)$ are center edges. The vertices e_1 and e_3, e_2 and e_4 are collinear with the vertex v.

Definition 5. A symmetric zonal mesh is defined as a sequence, consisting of symmetric polygon groups or joint polygon groups, with the property that every two adjacent groups share an edge that intersects at the center edge. The polygon consisting of center edges is called center-edge polygon. Fig. 4 shows some examples of this definition.

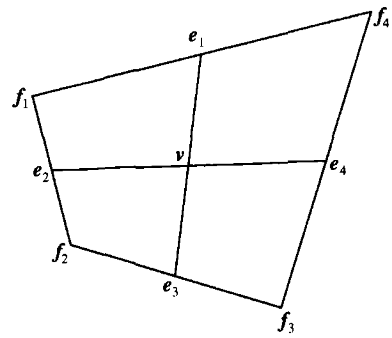


Fig. 3. Joint polygon group.

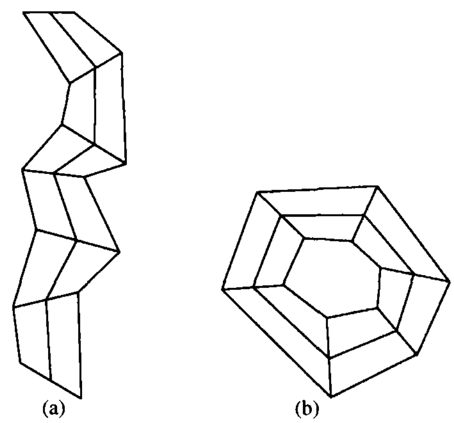


Fig. 4. Symmetric zonal mesh. The zonal mesh in (a) is open; in (b) is closed.

Based on the definitions mentioned above, we introduce several lemmas firstly.

Lemma 1. Let G be one of the symmetric polygon groups of a symmetric zonal mesh. The new edge points of G's center edges are the same as the new control points after one application of knot insertion step on center-edge polygon, which defines a uniform cubic B-spline curve.

For convenience, Fig. 5 will be in aid of the proof. In Fig.5, the vertices v, e_i and $f_i (i = 1, 2, 3, 4)$ are vertices of the symmetric polygon group G, and the vertices v^1, e_i^1 and $f_i^1 (i = 1, 2, 3, 4)$ are new vertices after one Catmull-Clark subdivision step on G.

Proof. By Definition 2, we have

$$f_1 + f_2 = 2e_2, \quad f_4 + f_3 = 2e_4, \quad e_1 + e_3 = 2v. \tag{1}$$

By the rules of Catmull-Clark scheme, we have new face points, new edge points and new vertex points

$$f_i^1 = \frac{f_i + e_{i+1} + e_i + v}{4},$$

$$e_i^1 = \frac{v + e_i + f_{i-1}^1 + f_i^1}{4},$$

$$v^1 = \frac{1}{2}v + \frac{1}{4^2}\sum_{j=1}^{4} e_j + \frac{1}{4^2}\sum_{j=1}^{4} f_j^1, \qquad (2)$$

where subscripts are taken modulo 4.

Using the first equation of (2), we have

$$f_1^1 + f_2^1 = \frac{f_1 + f_2 + 2v + e_1 + 2e_2 + e_3}{4}.$$

Then substitute the first and third equations of (1) into the right side of the above equation, we obtain

$$f_1^1 + f_2^1 = v + e_2. \qquad (3)$$

Substituting Eq. (3) into the second equation of (2), where $i = 2$, we obtain

$$e_2^1 = \frac{v + e_2}{2}. \qquad (4)$$

In a similar way, we get

$$e_4^1 = \frac{v + e_4}{2}. \qquad (5)$$

Because

$$\sum_{j=1}^{4} e_j = 2v + e_2 + e_4, \quad \sum_{j=1}^{4} f_j^1 = 2v + e_2 + e_4,$$

we obtain

$$v^1 = \frac{3}{4}v + \frac{1}{8}(e_2 + e_4). \qquad (6)$$

Eqs. (4) ~ (6) give just the new control points after inserting the midpoints of the knot intervals one time.

Lemma 2. Let G be one of the symmetric polygon groups of a symmetric zonal mesh. The new edge points of non-center edges of G are symmetric about the new vertex point of the center vertex of G.

Proof. As Fig. 5 shows, we have to prove $2v^1 = e_1^1 + e_3^1$. Using the third equation of (1), the second equation of (2), and Eqs. (3) and (5), we have

$$e_1^1 + e_3^1 = v + \frac{1}{4}\sum_j f_j^1 = \frac{2}{3}v + \frac{1}{4}(e_2 + e_4) = 2v^1.$$

Lemma 3. After one application of the Catmull-Clark subdivision step on a symmetric polygon group, the mesh is still a symmetric polygon group.

Proof. The proof of this lemma follows the direct application of Lemmas 1 and 2.

The above results lead to the following theorem.

Theorem 1. Let M be a symmetric zonal mesh, and P the polygon consisting of all center edges of M. Then the limit surface of M interpolates the uniform cubic B-spline curve defined by the polygon P.

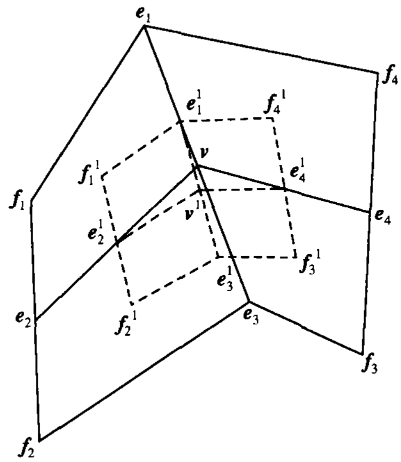


Fig. 5. Proof of Lemma 1.

Based on Theorem 1, we can obtain a corollary as follows.

Corollary 1. Let $\{M_i\}_{i=1}^m (m \leqslant 4)$ be m symmetric zonal meshes that have a common joint polygon group G_0. Denote by $\{c_i\}_{i=1}^m$ the corresponding curves interpolated by the limit surface of $\{M_i\}_{i=1}^m$. Then the curves $\{c_i\}_{i=1}^m$ meet at the center vertex of G_0, and (i) if the edges of M_i and M_j meeting G_0 are opposite, the curves c_i and c_j meet with C^2 continuity; (ii) if the edges of M_i and M_j meeting G_0 are adjacent, the curves c_i and c_j meet with C^0 continuity.

Proof. As showed in Fig. 6, by the definition of joint polygon group, we can know the vertices e_2, v and e_4 are collinear. So the cubic B-spline curve defined by short dashes in Fig. 6 is tangential with e_2v at the vertex v and also C^2 at the vertex v. Furthermore, because the vertices e_1, v and e_3 are collinear, the curve defined by long dashes is tangential with e_3v at the vertex v and also C^2 at the vertex v; but the vertices e_2, v and e_3 are not collinear, so the corresponding two curves are only C^0 at the vertex v.

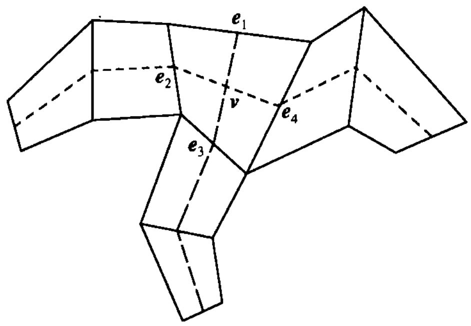
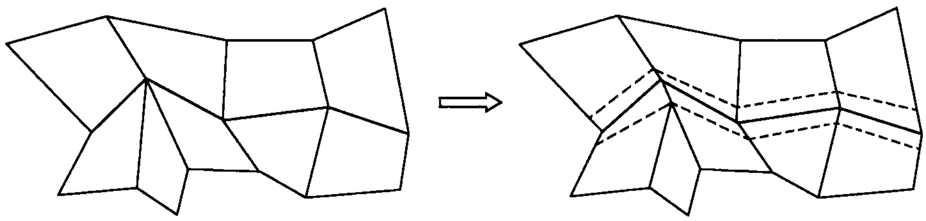


Fig. 6. Proof of Corollary 1.

Remark. We can construct a surface with sharp creases on some tagged edges by duplicating the vertices of the tagged edges on both sides of those edges. The duplication is only in the sense of the topology but not the geometric position. So the duplicated vertices have the same geometric position as their corresponding original vertices. The edges incident on the original vertices must be modified to be incident on the duplicated vertices. The vertices of the tagged edges and the duplicated vertices can compose a symmetric zonal mesh. So the limit subdivision surface can interpolate the curve defined by the tagged edges and is only C^0 along the curve (Fig. 7).



(a) (b)

Fig. 7. Duplicating vertices of the tagged edges. (a) The original polygonal mesh, thick solid lines are the tagged edges; (b) modified mesh, the vertices of dash lines are the duplicated vertices.

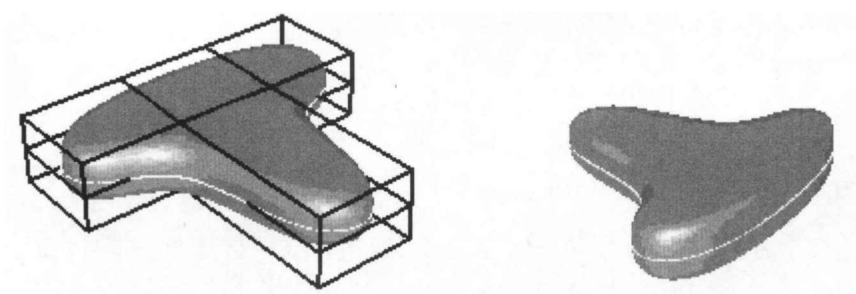
3 Curve interpolation

The problem dealt with in this paper can be described as follows.

Given several control polygons, denoted by $\{C_i\}_{i=1}^{l}$. We try to design a suitable polygonal mesh $\hat{M}$ so that the subdivision limit surface of $\hat{M}$ can interpolate the cubic B-spline curves $\{c_i\}_{i=1}^{l}$ defined by $\{C_i\}_{i=1}^{l}$. The number of curves intersecting at one vertex must be less than or equal four.

Applying Theorem 1, a symmetric zonal mesh, one for the control polygon of each given curve, can be constructed. Then the subdivision limit surface of the polygonal mesh including these symmetric zonal meshes is the satisfying surface interpolating the given curves.

From Fig. 8 to Fig. 12, we give some examples with our method.



(a) (b)

Fig. 8. The saddle. The black frame is the initial control polygon mesh of a saddle model, and the highlight curve is the interpolated curve.

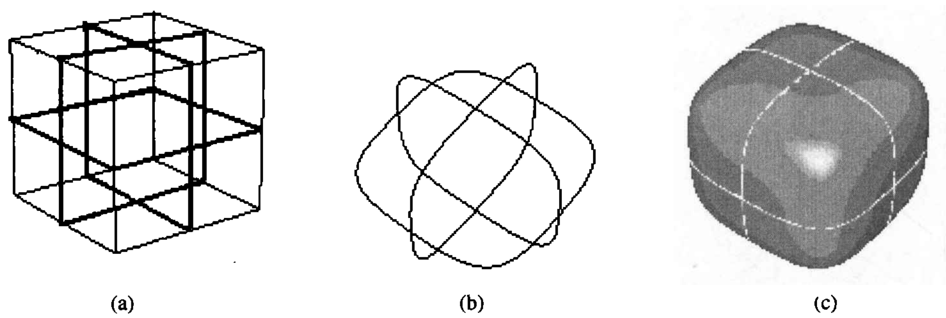


Fig. 9. Interpolating several intersecting curves. (a) A polygonal mesh consisting of symmetric zonal meshes, thick solid lines are the control polygons of the intersecting cubic B-spline curves; (b) a mesh consisting of the interpolated curves; (c) the interpolating surface, the highlight curves are the interpolated curves.

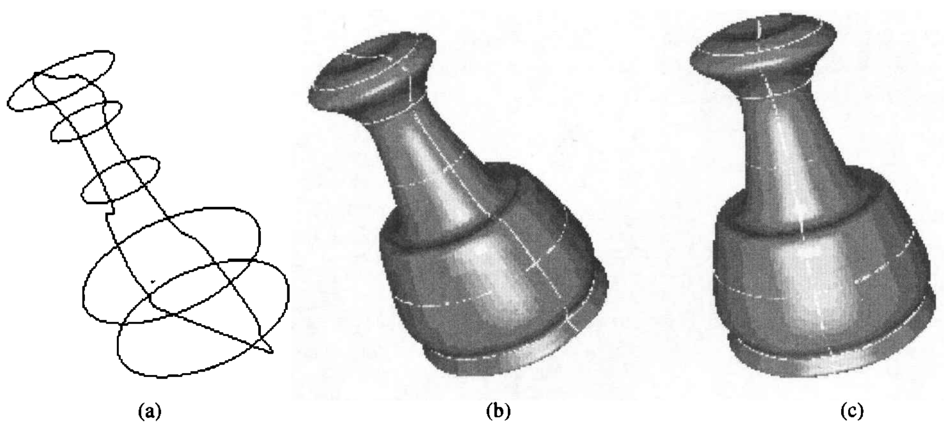


Fig. 10. (a) A curve mesh consisting of several interpolated curves; (b) and (c) the interpolating subdivision surface, the highlight curves are the interpolated curves.

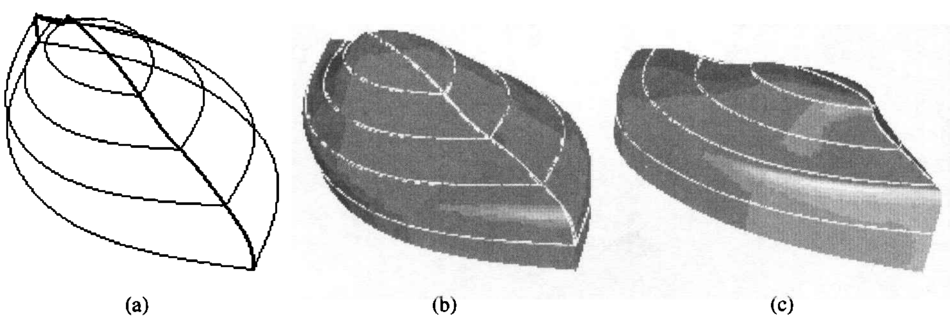
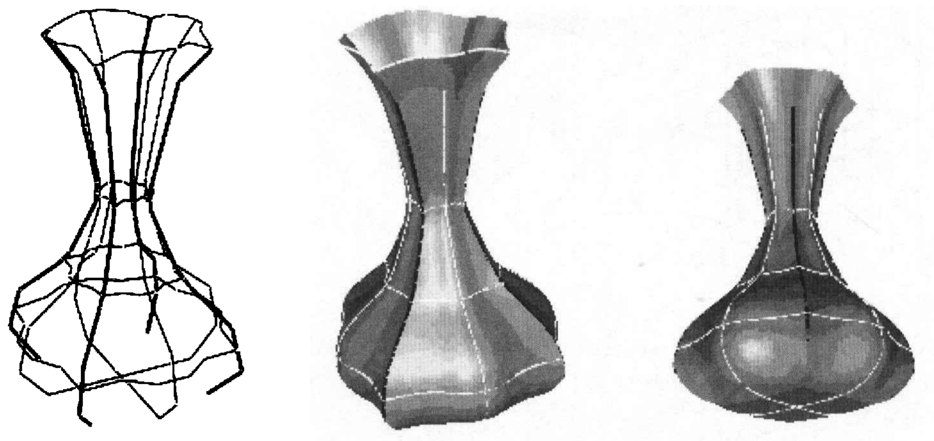


Fig. 11. A curve mesh consisting of several interpolated curves (a), the thick line curve is tagged where the limit interpolating subdivision surface (b, c) is only C^0.

4 Conclusion

This paper introduces a method for curve interpolation in the field of subdivision surface modeling. This sheme constructs a polygonal mesh containing "symmetric zonal meshes", which possesses some special properties. Since its limit interpolating surface's generation is based on the Catmull-Clark scheme, the surface is C^2 except for a finite number of points. Moreover sharp creases can also be modeled on the limit subdivision surface by duplicating the vertices of the tagged edges of initial mesh. The surface can interpolate and is only C^0 along the cubic B-spline curves that are defined by the tagged edges. The subdivision surface generated can interpolate a mesh of nonintersecting or intersecting curves that can be designed as feature lines or contour lines in the process of modeling. It is no doubt that our scheme makes Catmull-Clark subdivision scheme more attractive and more practical in surface modeling and com-



(a) (b) (c)

Fig. 12. The vase. (a) The mesh consisting of the control polygons of the interpolated curves, where the thick lines are tagged as sharp features; (b) and (c) the limit interpolating subdivision surface, the highlight curves are the interpolated curves and the black curves are the sharp feature curves where the surface is only C^0.

puter graphics.

One limitation of the proposed scheme is that the number of curves intersecting at one vertex must be no more than four. So our future work is to interpolate a curves mesh where more than four curves interpolate at one vertex.

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